

# HIRZEBRUCH $\chi_y$ -GENERA MODULO 8 OF FIBER BUNDLES FOR ODD INTEGERS $y$

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**ABSTRACT.** I. Hambleton, A. Korzeniewski and A. Ranicki have proved that the signature of a  $PL$  fibre bundle  $F \hookrightarrow E \rightarrow B$  is always multiplicative mod 4, i.e.  $\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{4}$ . In this paper we consider the Hirzebruch  $\chi_y$ -genera for odd integers  $y$  for a fiber bundle  $F \hookrightarrow E \rightarrow B$  such that  $E, F$  and  $B$  are smooth compact complex algebraic varieties. In particular, if  $y = 1$ , then  $\chi_1$  is the signature  $\sigma$ . We show that the Hirzebruch  $\chi_y$ -genera of such a fiber bundle are always multiplicative mod 4, i.e.  $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{4}$ . We also investigate multiplicativity mod 8, and show that if  $y \equiv 3 \pmod{4}$ , then  $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{8}$  and that in the case when  $y \equiv 1 \pmod{4}$  the Hirzebruch  $\chi_y$ -genera of such a fiber bundle is multiplicative mod 8 if and only if the signature is multiplicative mod 8, and that the non-multiplicativity modulo 8 is identified with an Arf-Kervaire invariant.

## 1. INTRODUCTION

The Hirzebruch  $\chi_y$ -genus was introduced by F. Hirzebruch [12] (also see [13]) in order to extend his famous Hirzebruch-Riemann-Roch theorem to a generalized one. If  $y = -1, 0, 1$ , then these  $\chi_y$ -genera are respectively

- $\chi_{-1}(X) = \chi(X)$  the *Euler-Poincaré characteristic*,
- $\chi_0(X) = \tau(X)$  the *Todd genus*,
- $\chi_1(X) = \sigma(X)$  the *signature*,

which are very important invariants in geometry and topology, and even in mathematical physics.

The Euler-Poincaré characteristic is multiplicative for *any* topological fiber bundle  $F \hookrightarrow E \rightarrow B$ , i.e.  $\chi(E) = \chi(F)\chi(B)$  holds. The signature is in general not multiplicative for fibre bundles. S. S. Chern, F. Hirzebruch and J.-P. Serre [9] proved that the signature is multiplicative for a fiber bundle under a certain monodromy condition, i.e. if the fundamental group  $\pi_1(B)$  of the base space  $B$  acts trivially on the cohomology group  $H^*(F; \mathbb{R})$  of the fiber space  $F$ . Later M. Atiyah [2], F. Hirzebruch [11] and K. Kodaira [14] gave the first examples of fibre bundles with non-multiplicative signatures.

I. Hambleton, A. Korzeniewski and A. Ranicki [10] showed that for a  $PL$  fibre bundle  $F \hookrightarrow E \rightarrow B$  of closed, connected, compatibly oriented  $PL$  manifolds

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{4}.$$

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In [18, 19] C. Rovi has shown that for a fiber bundle  $F \hookrightarrow E \rightarrow B$ , if the action of  $\pi_1(B)$  on  $H^m(F, \mathbb{Z})/\text{torsion} \otimes \mathbb{Z}_4$  is trivial (where  $\dim_{\mathbb{R}} F = 2m$ ),

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{8}.$$

In [18, 19] C. Rovi has also shown that the non-multiplicativity of the signature modulo 8 of a fibre bundle is detected the  $\mathbb{Z}_2$ -valued Arf-Kervaire invariant of a certain quadratic form associated to the fiber bundle.

In [20] S. Yokura has studied some explicit formulae of the Hirzebruch  $\chi_y$ -genera for complex fiber bundles  $F \hookrightarrow E \rightarrow B$  where  $F, E, B$  are smooth compact complex algebraic varieties, and he has observed that  $\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{4}$  just like the above result of Hambleton-Korzeniewski-Ranicki.

In this paper we consider such mod 4 and mod 8 multiplicativity formulae of Hirzebruch  $\chi_y$ -genera for the above complex algebraic fiber bundles  $F \hookrightarrow E \rightarrow B$ .

The main result of this paper is stated in Theorem 1.1, which gives an overview of the non-multiplicative behaviour of  $\chi_y$  genera modulo 8 of a fiber bundle for odd values of  $y$ . It is interesting to note that when  $y \equiv 1 \pmod{4}$  the  $\chi_y$ -genera adopt the same non-multiplicativity behaviour as the signature, and therefore the obstruction for multiplicativity in this case (when  $y \equiv 1 \pmod{4}$ ) is detected by the Arf-Kervaire invariant of a certain quadratic form associated to the fiber bundle, which is described in Theorem 1.1 below.

**Theorem 1.1.** *Let  $F \hookrightarrow E \rightarrow B$  be a fiber bundle such that  $F, E, B$  are smooth compact complex algebraic varieties.*

- (a) *If  $y \equiv 3 \pmod{4}$ , then  $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{8}$ .*
- (b) *If  $y \equiv 1 \pmod{4}$ , then  $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{8} \iff \sigma(E) \equiv \sigma(F)\sigma(B) \pmod{8}$ .*

*Moreover*

$$\chi_y(E) - \chi_y(F)\chi_y(B) \equiv 4\text{Arf}(W, \mu, h) \pmod{8}.$$

*where  $(W, \mu, h)$  is a certain  $\mathbb{Z}_2$ -valued quadratic form associated to the fiber bundle. (For details see Theorem 5.4 below.)*

The proof of Theorem 1.1 will be subdivided into shorter statements. In particular, (a) and the first part of (b) is proved in Theorem 5.1. The second part of (b), i.e. the identification of the obstruction to multiplicativity with the Arf-Kervaire invariant is shown in Theorem 5.4.

There is another result which follows from [18, Theorem 3.5] and Theorem 1.1 :

**Theorem 1.2.** *Let  $F \hookrightarrow E \rightarrow B$  be a fiber bundle such that  $F, E, B$  are smooth compact complex algebraic varieties with  $\dim_{\mathbb{R}} F = 2m$ . If the action of  $\pi_1(B)$  on  $H^m(F, \mathbb{Z})/\text{torsion} \otimes \mathbb{Z}_4$  is trivial, then for any odd integer  $y$*

$$\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{8}.$$

**Remark 1.3.** From Theorem 1.2 we can immediately observe that *if  $B$  is simply connected*, such as smooth complex rational varieties and smooth Fano varieties, then for *any* odd integer  $y$

$$\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{8}.$$

However, in this case the usual equality does hold:

$$\chi_y(E) = \chi_y(F)\chi_y(B) \in \mathbb{Z}.$$

This is because in this case the action of  $\pi_1(B)$  is automatically trivial on  $H^*(F, \mathbb{Z})$ , from which  $\chi_y(E) = \chi_y(F)\chi_y(B)$  follows, as explained in [15, Remark 4.6] (also see [6, 7, 8, 15]).

## 2. HIRZEBRUCH $\chi_y$ -GENERA

First we recall the definition of the Hirzebruch  $\chi_y$ -genus. Let  $X$  be a smooth complex algebraic variety. The  $\chi_y$ -genus of  $X$  is defined by

$$\chi_y(X) := \sum_{p \geq 0} \chi(X, \Lambda^p T^* X) y^p = \sum_{p \geq 0} \left( \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(X, \Lambda^p T^* X) \right) y^p.$$

Thus the  $\chi_y$ -genus is the generating function of the Euler-Poincaré characteristic  $\chi(X, \Lambda^p T^* X)$  of the sheaf  $\Lambda^p T^* X$ , which shall be simply denoted by  $\chi^p(X)$ :

$$\chi_y(X) = \sum_{p \geq 0} \chi^p(X) y^p.$$

Since  $\Lambda^p T^* X = 0$  for  $p > \dim_{\mathbb{C}} X$ ,  $\chi_y(X)$  is a polynomial of at most degree  $\dim_{\mathbb{C}} X$ . Note that for an ordinary product of spaces  $\chi_y$  is multiplicative, i.e.  $\chi_y(X \times Y) = \chi_y(X)\chi_y(Y)$ .

Then we have the following “generalized Hirzebruch-Riemann-Roch theorem” (abbr., gHRR):

$$\chi_y(X) = \int_X T_y(TX) \cap [X] \in \mathbb{Q}[y],$$

where  $T_y(TX)$  is the generalized Todd class of the tangent bundle of  $X$ .  $T_y(TX)$  which we will denote simply by  $T_y(X)$  is defined as follows:

$$T_y(X) := \prod_{i=1}^{\dim X} \left( \frac{\alpha_i(1+y)}{1 - e^{-\alpha_i(1+y)}} - \alpha_i y \right),$$

where  $\alpha_i$  are the Chern roots of the tangent bundle  $TX$ , i.e.,

$$c(X) = \prod_{i=1}^{\dim X} (1 + \alpha_i).$$

Note that the normalized power series

$$Q_y(\alpha) := \frac{\alpha(1+y)}{1 - e^{-\alpha(1+y)}} - \alpha y \in \mathbb{Q}[y][[\alpha]].$$

specializes to

$$Q_{-1}(\alpha) = 1 + \alpha, \quad Q_0(\alpha) = \frac{\alpha}{1 - e^{-\alpha}}, \quad Q_1(\alpha) = \frac{\alpha}{\tanh \alpha}$$

Therefore  $T_y(X)$  unifies the following important characteristic cohomology classes of  $X$ :

$$c(X) = \prod_{i=1}^{\dim X} (1 + \alpha_i), \quad td(X) = \prod_{i=1}^{\dim X} \frac{\alpha_i}{1 - e^{-\alpha_i}}, \quad L(X) = \prod_{i=1}^{\dim X} \frac{\alpha_i}{\tanh \alpha_i},$$

which are respectively the *Chern class*, *Todd class* and *L-class*.  $T_y(X)$  can be considered as a parameterized Todd class  $td(X)$  by  $y$ . We call this parameterized Todd class  $T_y(X)$  the *Hirzebruch class* of  $X$ .

For the distinguished three values  $-1, 0, 1$  of  $y$ , by the definition we have the following:

- the Euler-Poincaré characteristic:

$$\chi(X) = \chi_{-1}(X) = \chi^0(X) - \chi^1(X) + \chi^2(X) - \cdots + (-1)^n \chi^n(X),$$

- the Todd genus:

$$\tau(X) = \chi_0(X) = \chi^0(X),$$

- the signature:

$$\sigma(X) = \chi_1(X) = \chi^0(X) + \chi^1(X) + \chi^2(X) + \cdots + \chi^n(X).$$

As noted by Hirzebruch in [12, §15.5] the following duality formula holds

$$(2.1) \quad \chi^p(X) = (-1)^n \chi^{n-p}(X).$$

This duality formula will play an important role in this paper.

### 3. MULTIPLICATIVITY MOD 2

First we give the following definitions:

**Definition 3.1.** Let  $\chi^p(X) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(X, \Lambda^p T^* X)$ . Then

$$\chi^{\text{odd}}(X) = \chi^1(X) + \chi^3(X) + \chi^5(X) \cdots \quad \text{is the odd part.}$$

$$\chi^{\text{even}}(X) = \chi^0(X) + \chi^2(X) + \chi^4(X) \cdots \quad \text{is the even part.}$$

They shall be respectively called the *even  $\chi$ -genus* (or the *even  $\chi$ -characteristic*) and the *odd  $\chi$ -genus* (or the *odd  $\chi$ -characteristic*).

Using the formulas for the Euler characteristic and the signature in terms of  $\chi^p(X)$  we note that

$$\chi(X) = \chi^{\text{even}}(X) - \chi^{\text{odd}}(X),$$

$$\sigma(X) = \chi^{\text{even}}(X) + \chi^{\text{odd}}(X).$$

Hence we have

$$(3.2) \quad \sigma(X) + \chi(X) = 2\chi^{\text{even}}(X),$$

$$(3.3) \quad \sigma(X) - \chi(X) = 2\chi^{\text{odd}}(X),$$

Thus from either (3.2) or (3.3) we can deduce two well-known facts about the Euler characteristic and the signature modulo 2:

**Corollary 3.4** (mod 2 formula). *For any smooth compact complex algebraic variety  $X$  we have*

$$\sigma(X) \equiv \chi(X) \pmod{2}.$$

**Corollary 3.5.** *If  $n = \dim_{\mathbb{C}} X$  is odd, then  $\chi(X) \equiv 0 \pmod{2}$ , i.e.  $\chi(X)$  is even.*

This is because  $\sigma(X) = 0$  for dimension reasons, since  $\dim_{\mathbb{R}} X = 2n = 4k + 2$  (letting  $n = 2k + 1$ ).

**Corollary 3.6.** *For any fiber bundle  $F \hookrightarrow E \rightarrow B$  with  $F, E, B$  smooth compact complex algebraic varieties, we have*

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{2}.$$

*Proof.* Indeed, using (3.2), we have  $\sigma(E) + \chi(E) - (\sigma(F \times B) + \chi(F \times B)) \equiv 0 \pmod{2}$ , which becomes  $\sigma(E) - \sigma(F)\sigma(B) \equiv 0 \pmod{2}$ , since  $\sigma(F \times B) = \sigma(F)\sigma(B)$  and  $\chi(E) = \chi(F \times B)$ . Thus we get the result.  $\square$

As we will see in Theorem 4.15, the  $\chi_y$  genera of a fibre bundle with  $y$  odd are multiplicative modulo 4, therefore multiplicativity also holds modulo 2,

$$\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{2}$$

for all odd integers  $y$ .

#### 4. MULTIPLICATIVITY MOD 4

Let  $\dim_{\mathbb{C}} X = 2n$ . Now using the duality formula (2.1) we get the following:

$$\begin{aligned} \chi_y(X) &= \sum_{i=0}^{n-1} \chi^i(X)(y^i + y^{2n-i}) + \chi^n(X)y^n \\ &= \sum_{i=0}^{n-1} \chi^i(X)y^i(1 + y^{2n-2i}) + \chi^n(X)y^n. \end{aligned}$$

Thus we have

$$\begin{aligned} \chi(X) &= \chi_{-1}(X) = \sum_{i=0}^{n-1} (-1)^i 2\chi^i(X) + (-1)^n \chi^n(X), \\ \sigma(X) &= \chi_1(X) = \sum_{i=0}^{n-1} 2\chi^i(X) + \chi^n(X). \end{aligned}$$

$$(4.1) \quad \sigma(X) + \chi(X) = 2 \sum_{i=0}^{n-1} (1 + (-1)^i) \chi^i(X) + (1 + (-1)^n) \chi^n(X),$$

$$(4.2) \quad \sigma(X) - \chi(X) = 2 \sum_{i=0}^{n-1} \left(1 + (-1)^{i+1}\right) \chi^i(X) + \left(1 + (-1)^{n+1}\right) \chi^n(X).$$

In the case when  $n = 2k$ , we have

$$(4.3) \quad \sigma(X) + \chi(X) = 4 \sum_{j=0}^{k-1} \chi^{2j}(X) + 2\chi^{2k}(X),$$

$$(4.4) \quad \sigma(X) - \chi(X) = 4 \sum_{j=1}^k \chi^{2j-1}(X).$$

(4.4) implies the following:

**Corollary 4.5.** *For any smooth compact complex algebraic variety of complex dimension  $2n$  with an even integer  $n$*

$$\sigma(X) \equiv \chi(X) \pmod{4}.$$

**Corollary 4.6.** *For any fiber bundle  $F \hookrightarrow E \rightarrow B$  such that  $\dim_{\mathbb{C}} E = 2n$  with an even integer  $n$ , we have*

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{4}.$$

The proof is just like the above case of mod 2 formula.

Next, we consider the case when  $n = 2k + 1$ . It follows from (4.1) and (4.2) that we have

$$(4.7) \quad \sigma(X) + \chi(X) = 4 \sum_{j=0}^k \chi^{2j}(X)$$

$$(4.8) \quad \sigma(X) - \chi(X) = 4 \sum_{j=1}^k \chi^{2j-1}(X) + 2\chi^{2k+1}(X).$$

Hence from (4.7) we obtain the following

**Corollary 4.9.** *For any smooth compact complex algebraic variety of complex dimension  $2n$  with an odd integer  $n$*

$$\sigma(X) + \chi(X) \equiv 0 \pmod{4}.$$

**Remark 4.10.** We should note that in the case when  $n$  is an odd integer, we do not have

$$\sigma(X) \equiv \chi(X) \pmod{4}.$$

However we do have:

**Corollary 4.11.** *For any fiber bundle  $F \hookrightarrow E \rightarrow B$  such that  $\dim_{\mathbb{C}} E = 2n$  with an odd integer  $n$ ,*

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{4}.$$

The proof is just like the above, using Equation (4.7).

**Remark 4.12.** Let  $F \hookrightarrow E \rightarrow B$  be a fiber bundle such that  $\dim_{\mathbb{C}} E = 2n$ . We have shown that for any integer  $n$  we have  $\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{4}$ . Here, using the following expressions

$$\sum_{j=0}^k \chi^{2j-1}(X) = \frac{\chi^{\text{odd}}(X)}{2}, \quad \sum_{j=0}^k \chi^{2j}(X) = \frac{\chi^{\text{even}}(X)}{2},$$

we can observe the following:

(1) If  $n$  is even,

$$\sigma(E) - \sigma(F)\sigma(B) = 4 \left( \frac{\chi^{\text{odd}}(E) - \chi^{\text{odd}}(F \times B)}{2} \right),$$

and we have

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{8} \iff \chi^{\text{odd}}(E) \equiv \chi^{\text{odd}}(F \times B) \pmod{4}.$$

(2) If  $n$  is odd,

$$\sigma(E) - \sigma(F)\sigma(B) = 4 \left( \frac{\chi^{\text{even}}(E) - \chi^{\text{even}}(F \times B)}{2} \right)$$

and we have

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{8} \iff \chi^{\text{even}}(E) \equiv \chi^{\text{even}}(F \times B) \pmod{4}.$$

Now we will discuss multiplicativity mod 4 of  $\chi_y$ -genera of a fiber bundle of smooth, compact complex algebraic varieties. The following congruence is crucial for both the mod 4 result and for the mod 8 result in the next section.

**Proposition 4.13.**

$$\chi_y(X) \equiv \frac{\sigma(X)}{2}(1+y) + \frac{\chi(X)}{2}(1-y) \pmod{1-y^2}.$$

*Proof.* Considering mod  $1-y^2$  means “letting  $y^2 = 1$  in the polynomial  $\chi_y(X)$ ”:

$$\begin{aligned} \chi_y(X) &= \sum \chi^i(X) y^i \\ &\equiv \chi^0(X) + \chi^1(X)y + \chi^2(X) + \chi^3(X)y + \chi^4(X) + \chi^5(X)y + \cdots \pmod{1-y^2} \\ &= \chi^{\text{even}}(X) + \chi^{\text{odd}}(X)y \pmod{1-y^2} \\ &= \frac{\sigma(X) + \chi(X)}{2} + \frac{\sigma(X) - \chi(X)}{2}y \pmod{1-y^2} \\ &= \frac{\sigma(X)}{2}(1+y) + \frac{\chi(X)}{2}(1-y) \pmod{1-y^2} \end{aligned}$$

□

**Remark 4.14.** The polynomial  $1-y^2$  is zero at  $y = -1, 1$ , for which we have the special values  $\chi_{-1}(X) = \chi(X)$  and  $\chi_1(X) = \sigma(X)$ .

Now we are ready to prove the following

**Theorem 4.15.** *Let  $F \hookrightarrow E \rightarrow B$  be a fiber bundle such that  $F, E, B$  are smooth compact complex algebraic varieties. Let  $y$  be an odd integer, then*

$$\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{4}.$$

*Proof.* From 4.13, we can see that for any fiber bundle  $F \hookrightarrow E \rightarrow B$  we have

$$\chi_y(E) - \chi_y(F)\chi_y(B) \equiv \frac{\sigma(E) - \sigma(F)\sigma(B)}{2}(1+y) \pmod{1-y^2}.$$

We have shown before that  $\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{4}$ , i.e.  $\sigma(E) - \sigma(F)\sigma(B)$  is divisible by 4, thus  $\frac{\sigma(E) - \sigma(F)\sigma(B)}{2}$  is even. Since  $y$  is odd,  $1+y$  is even, thus  $\frac{\sigma(E) - \sigma(F)\sigma(B)}{2}(1+y) \equiv 0 \pmod{4}$ , and  $1-y^2 = (1-y)(1+y) \equiv 0 \pmod{4}$ . Thus we have  $\chi_y(E) - \chi_y(F)\chi_y(B) \equiv 0 \pmod{4}$ , i.e. for any odd integer  $y$  we have

$$\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{4}.$$

□

## 5. MULTIPLICATIVITY MOD 8

We will now investigate multiplicativity modulo 8 and prove the main results of this note mentioned in the introduction.

**Theorem 5.1.** *Let  $F \hookrightarrow E \rightarrow B$  be a fiber bundle such that  $F, E, B$  are smooth compact complex algebraic varieties. Let  $y$  be an odd integer, then*

- (1) *If  $y \equiv 3 \pmod{4}$ , then  $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{8}$ .*
- (2) *If  $y \equiv 1 \pmod{4}$ , then  $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{8} \iff \sigma(E) \equiv \sigma(F)\sigma(B) \pmod{8}$ .*

*Proof.* Again using 4.13, we can see that for any fiber bundle  $F \hookrightarrow E \rightarrow B$  we have

$$\chi_y(E) - \chi_y(F)\chi_y(B) \equiv \frac{\sigma(E) - \sigma(F)\sigma(B)}{2}(1+y) \pmod{1-y^2}.$$

- (1) First we observe that if  $y$  is odd, then in fact we have  $1-y^2 \equiv 0 \pmod{8}$ . Indeed, let  $y = 2k+1$ . Then  $1-y^2 = (1-y)(1+y) = -2k(2k+2) = -2 \cdot 2 \cdot k(k+1)$  is divisible by  $2 \cdot 2 \cdot 2 = 8$  because  $k(k+1)$  is always even. Let  $y \equiv 3 \pmod{4}$ , i.e.  $1+y$  is divisible by 4. Then we have  $\chi_y(E) - \chi_y(F)\chi_y(B) \equiv \frac{\sigma(E) - \sigma(F)\sigma(B)}{2}(1+y) \equiv 0 \pmod{8}$ . Therefore we have

$$y \equiv 3 \pmod{4} \implies \chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{8}.$$

- (2) Let  $y \equiv 1 \pmod{4}$ , i.e. let  $y = 4k+1$ . Then  $1+y = 4k+2 = 2(2k+1)$ . Thus

$$\chi_y(E) - \chi_y(F)\chi_y(B) \equiv \frac{\sigma(E) - \sigma(F)\sigma(B)}{2}(1+y) \equiv 0 \pmod{8}$$

if and only if

$$\frac{\sigma(E) - \sigma(F)\sigma(B)}{2} \equiv 0 \pmod{4}.$$



I.e.,

$$\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{8} \iff \sigma(E) \equiv \sigma(F)\sigma(B) \pmod{8}$$

□

**Remark 5.2.** If we consider  $\chi_y(X)$  modulo  $y - y^3 = y(1 - y)(1 + y)$ , then we have

$$\chi_y(X) \equiv \tau(X)(1 - y^2) + \frac{\chi(X)}{2}(y^2 - y) + \frac{\sigma(X)}{2}(y^2 + y) \pmod{y - y^3}.$$

Indeed, considering mod  $y - y^3$  means “letting  $y^3 = y$  in the polynomial  $\chi_y(X)$ ”, thus we have

$$\begin{aligned} \chi_y(X) &= \sum \chi^i(X)y^i \\ &\equiv \chi^0(X) + \chi^1(X)y + \chi^2(X)y^2 + \chi^3(X)y + \chi^4(X)y^2 + \chi^5(X)y + \cdots \pmod{y - y^3} \\ &= \chi^0(X) + \chi^{\text{odd}}(X)y + (\chi^{\text{even}}(X) - \chi^0(X))y^2 \pmod{y - y^3} \\ &= \tau(X) + \frac{\sigma(X) - \chi(X)}{2}y + \left( \frac{\sigma(X) + \chi(X)}{2} - \tau(X) \right)y^2 \pmod{y - y^3} \\ &= \tau(X)(1 - y^2) + \frac{\chi(X)}{2}(y^2 - y) + \frac{\sigma(X)}{2}(y^2 + y) \pmod{1 - y^2} \end{aligned}$$

In this case we have the following formula:

$$\chi_y(E) - \chi_y(F)\chi_y(B) \equiv (\tau(E) - \tau(F)\tau(B))(1 - y^2) + \frac{\sigma(E) - \sigma(F)\sigma(B)}{2}(y^2 + y) \pmod{y - y^3}.$$

Note that  $y - y^3$  has zeros at  $y = 0, -1, 1$  and we have  $\chi_0(X) = \tau(X)$ ,  $\chi_{-1}(X) = \chi(X)$ ,  $\chi_1(X) = \sigma(X)$ .

In Theorem 5.1 (2) we have shown that when  $y \equiv 1 \pmod{4}$ , the  $\chi_y$ -genera of a fiber bundle have the same multiplicativity behaviour as the signature modulo 8. This means that we can use the results about the multiplicativity of signature modulo 8 from [18, 19] to prove statements for  $\chi_y$ -genera with  $y \equiv 1 \pmod{4}$ . The first of these statements involves the definition of the Arf invariant of a certain  $\mathbb{Z}_2$ -quadratic form associated to the fiber bundle. For the convenience of the reader we recall here some relevant definitions and give precise references of where some necessary proofs can be found.

We start by defining a non-singular quadratic form over  $\mathbb{Z}_2$  and the Arf invariant. Let  $V$  be a  $\mathbb{Z}_2$ -vector space and  $\lambda$  a non-singular symmetric bilinear form

$$\lambda : V \otimes V \rightarrow \mathbb{Z}_2,$$

and let  $h : V \rightarrow \mathbb{Z}_2$  be a  $\mathbb{Z}_2$ -valued quadratic enhancement of this bilinear form which satisfies the following property,

$$h(x + y) = h(x) + h(y) + \lambda(x, y) \in \mathbb{Z}_2.$$

The Arf invariant was first defined in [1] as follows,

**Definition 5.3.** With a symplectic basis  $\{e_1, \dots, e_k, \bar{e}_1, \dots, \bar{e}_k\}$  for  $V$ , the Arf invariant is defined as

$$\text{Arf}(h) = \sum_{j=1}^k h(e_j)h(\bar{e}_j) \in \mathbb{Z}_2.$$

A *characteristic element* of a symmetric form  $(V, \lambda)$  is an element such that for any  $u \in V$  it holds that

$$\lambda(u, u) = \lambda(u, v) \in \mathbb{Z}_2.$$

For example the Wu class  $v_{2k}(M) \in H^{2k}(M; \mathbb{Z}_2)$  of a  $4k$ -dimensional manifold  $M$  is a characteristic element of the intersection form.

A *sublagrangian subspace* of a symmetric form  $(V, \lambda)$  is a subspace  $L$  such that  $\lambda(L, L) = 0$ .

A *Lagrangian subspace* of a symmetric form  $(V, \lambda)$  is a subspace  $L$  such that  $\lambda(L, L) = 0$  and  $\dim(L) = (\frac{1}{2})\dim(V)$ .

The relation between the signature modulo 8 of a manifold and the Arf invariant was investigated in [18, Proposition 2.4.5 and Theorem 4.3.5.]. This relation is intricately related to the  $\mathbb{Z}_8$ -valued Brown-Kervaire invariant defined in [4] and Morita's theorem [16, Theorem 1.1]. Morita's theorem requires the use of the *Pontryagin squares*  $\mathcal{P}_2$ . A good reference for this is [17, Chapter 2]. The relation states that when the signature of a  $4k$ -dimensional manifold is divisible by 4, then modulo 8 this signature can be expressed as 4 times the Arf-Kervaire invariant of an associated  $\mathbb{Z}_2$ -valued quadratic form. The details about how to construct this associated quadratic form are given in [18, Proposition 2.4.5 and Theorem 4.3.5.] and the construction for the case of a fiber bundle is given in Theorem 5.4.

**Theorem 5.4.** Let  $F \hookrightarrow E \rightarrow B$  be a fiber bundle such that  $F, E, B$  are smooth compact complex algebraic varieties. If  $y \equiv 1 \pmod{4}$ , then

$$\chi_y(E) - \chi_y(F)\chi_y(B) \equiv 4\text{Arf}(W, \mu, h) \pmod{8}.$$

where  $(W, \mu, h) = (L^\perp/L, [\lambda \oplus -\lambda'], [\mathcal{P}_2 \oplus -\mathcal{P}'_2]/2)$  with

- $L = \langle v_{2k} \rangle \subset L^\perp$ , with  $v_{2k} = v_{2k}(E) \oplus v_{2k}(F \times B) \in H^{2k}(E; \mathbb{Z}_2) \oplus H^{2k}(F \times B; \mathbb{Z}_2)$  the Wu class of  $E \sqcup F \times B$
- $L^\perp = \{(x, x') \in H^{2k}(E; \mathbb{Z}_2) \oplus H^{2k}(F \times B; \mathbb{Z}_2) \mid \lambda(x, x) = \lambda'(x', x') \in \mathbb{Z}_2\}$
- $\mathcal{P}_2$  and  $\mathcal{P}'_2$  are the Pontryagin squares of  $E$  and  $F \times B$  respectively.

*Proof.* This theorem is a direct consequence of [18, Theorem 6.2.1] and Theorem 5.1. □

The other statement which follows from the work on the signature modulo 8 in [18] and [19] is as follows:

**Theorem 5.5.** Let  $F \hookrightarrow E \rightarrow B$  be a fiber bundle such that  $F, E, B$  are smooth compact complex algebraic varieties with  $\dim_{\mathbb{R}} F = 2m$ . If the action of  $\pi_1(B)$  on  $H^m(F, \mathbb{Z})/\text{torsion} \otimes \mathbb{Z}_4$  is trivial, then for any odd integer  $y$

$$\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{8}.$$

*Proof.* In [18, Theorem 6.3.1] Rovi showed that if the action of  $\pi_1(B)$  on  $H^m(F, \mathbb{Z})/\text{torsion} \otimes \mathbb{Z}_4$  is trivial, then

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{8}.$$

For a more succinct proof of this result see [19, Theorem 3.5]. Combining this with Theorem 5.1 (2) we obtain

$$\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{8}.$$

□

## 6. SOME REMARKS

**Remark 6.1.** As to the congruence formulae for  $\chi_y$ , above we consider the case when  $y$  is an odd integer. When it comes to the case when  $y$  is an even integer, we have not found any interesting congruence formula.

**Remark 6.2.** In [8] Cappell-Libgober-Maxim-Shaneson obtain the following Atiyah-Meyer type formula:

$$\chi_y(E) = \int_B ch^*(\chi_y(\pi)) \cup \tilde{T}_y^*(TB),$$

where  $ch^*$  is the Chern character,  $\chi_y(\pi)$  is the  $K$ -theory  $\chi_y$ -characteristic of the bundle projection map  $\pi : E \rightarrow B$  and  $\tilde{T}_y^*(TB)$  is the unnormalized Hirzebruch class. For more detailed explanation of these see [8]. Since  $ch^0(\chi_y(\pi)) = \chi_y(F)$ , as explained in [8], the right-hand-side of the above Atiyah-Meyer type formula is

$$\int_B ch^*(\chi_y(\pi)) \cup \tilde{T}_y^*(TB) = \chi_y(F)\chi_y(B) + \text{the other terms}.$$

Hence it follows from our results above that

- (1) for any odd integer  $y$  the other terms is divisible by 4,
- (2) if  $y \equiv 3 \pmod{4}$ , the other terms is divisible by 8,
- (3) if  $y \equiv 1 \pmod{4}$ , the other terms is divisible by 8 if and only if  $\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{8}$ .

It remains to be seen if one could get the above results directly from the above Atiyah-Meyer type formula.

**Remark 6.3.** Even if  $X$  is singular, we can define  $\chi_y(X)$  (using the same symbol) using the mixed Hodge structure (e.g. see [3, 6, 8, 15]). In this case we can consider the above congruences even for fiber bundles  $F \hookrightarrow E \rightarrow B$  with  $F, E, B$  being possibly singular. In this case we have the following results, mod 4 and mod 8 being replaced by respectively mod 2 and mod 4:

- (1) For any odd integer  $y$ ,  $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{2}$ .
- (2) If  $y \equiv 3 \pmod{4}$ , then  $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{4}$ .
- (3) If  $y \equiv 1 \pmod{4}$ , then  $\chi_y(E) \equiv \chi_y(F)\chi_y(B) \pmod{4} \iff \sigma(E) \equiv \sigma(F)\sigma(B) \pmod{4}$ .

However, if the duality formula (2.1) in §2 still holds even in the singular case, then the same results as in the smooth case hold.

**Remark 6.4.** In this paper the congruence formula (4.13) of two integral polynomials is a key. For  $a(y), b(y) \in \mathbb{Z}[y]$ , the congruence  $a(y) \equiv 0 \pmod{b(y)}$  of course means that  $\exists c(y) \in \mathbb{Z}[y]$  such that  $a(y) = b(y)c(y)$ . Then for any integer  $n \in \mathbb{Z}$  we have  $a(n) = b(n)c(n)$ , i.e.  $a(n) \equiv 0 \pmod{b(n)}$ . (In our case we consider only odd integers  $n$ , though.) Namely, we have

$$a(y) \equiv 0 \pmod{b(y)} \text{ (in } \mathbb{Z}[y]) \implies \forall n \in \mathbb{Z}, a(n) \equiv 0 \pmod{b(n)} \text{ (in } \mathbb{Z}).$$

It should be noted that the converse of this implication does not necessarily hold. A counterexample is given by Fermat's little theorem. Indeed, let  $p$  be a prime number and  $a(y) = y^p - y, b(y) = p$ . Fermat's Little Theorem says that for any integer  $n \in \mathbb{Z}$   $n^p \equiv n \pmod{p}$ , i.e. for  $\forall n \in \mathbb{Z}$   $a(n) = n^p - n \equiv 0 \pmod{p}$ . However clearly  $a(y) = y^p - y \not\equiv 0 \pmod{p}$  (in  $\mathbb{Z}[y]$ ). The case when  $p = 5$  is pointed out in [5], where L.F. Cáceres and J. A. Vélez-Marulanda consider some special cases when this converse does hold.

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